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CONTENTS

Algebra

Analysis

(Continues on inside back cover)

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Felelős szerkesztő: Kérchy László

FYX Bt., Szeged – Felelős vezető: Kurusa Árpád ügyvezető

(Continued from back cover)

Geometry

Probability Theory

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Isomorphisms of matrix groups over commutative rings

V. M. Petechuk and J. V. Petechuk

Communicated by M. B. Szendrei

Abstract. We give a description of the isomorphism classes of matrix groups over commutative rings with 1 and that have dimension more than 3 and containing the group of elementary transvections. We characterize those homomorphisms of matrix groups, which satisfy the so-called $(*)$ condition. Such homomorphisms can be constructed with the help of the standard homomorphism. We apply the characterization obtained to the description of the above class of matrix groups.

1. Introduction

The study of the group of automorphisms of classical matrix groups was started by the paper [28], in which the authors described the group of automorphisms of the group $PSL(n, R)$ over an arbitrary field R for $n \geq 3$. Extending the ideas of [16], later J. Dieudonné [7] and C. E. Rickart [27] introduced the method of involutions, using which they described the group of automorphisms of the group $GL(n, R)$ over a skew-field R for $n \geq 3$.

The first step in building the theory of automorphisms over rings, particulary for the group $GL(n, \mathbb{Z})$ over the ring of integers $\mathbb{Z}$ with $n \geq 3$, was done in [13]. In [14] the authors generalized Hua–Reiner's results over non-commutative principal domains.

The methods of the papers above were based in general on the study of properties of involutions in these groups considered. O'Meara (1966–1970) came up with a completely new so-called method of residual spaces, which does not use

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involutions. Using this method he managed to describe the group of automorphisms of the group $GL(n, R)$ for $n \geq 3$.

Independently of O'Meara, based on the study of involutions, the group of automorphisms of the group $E(n, R)$ over an integer domain R of characteristic different from 2 for $n \geq 3$ was described by Shi-Jian Yan (see [29]). In [26] was defined the group of automorphisms of the group $GL(n, R)$ over a commutative local ring R where 2 is an invertible in R for $n \geq 3$. They used Kaplansky's theorem which claims that the projective modules over a local ring are free. Note that the invertibility of the element 2 gives the possibility to draw upon the study of the automorphisms of the group $GL(n, R)$ for $n \geq 3$, the technique which leans on the study of involutions.

W. Waterhouse in [32] proved that all automorphisms of the group $GL(n, R)$ with $n \geq 3$ over an arbitrary commutative ring R in which 2 is invertible are standard. If 2 is not an invertible element of a commutative local ring R, then the automorphisms of the groups $E(n, R)$ and $GL(n, R)$ were described in [18–20]. V. Petechuk was first to discover that in some cases a nonstandard automorphism exists when $n = 3$, through which he obtained a full description of the automorphisms of the group $E(n, R)$.

In [19] these results were carried over to arbitrary commutative rings. Using different methods in [10, 21, 22, 34] were given a full description of the automorphisms of the group $E(n, R)$ over arbitrary associative rings with 1 in which 2 is an invertible element and $n \geq 3$.

I. Golubchik in [9] described the isomorphisms of matrix groups $GL(n, R)$ for $n \geq 3$ and $GL(m, K)$ for $m \geq 4$ over arbitrary associative rings R and K with 1. The homomorphisms with condition $(*)$ which is mentioned in this paper were described in [24, 25]. The homomorphisms with condition $(*)$ are in some sense connected with the theory of representation of matrix groups over rings (for example, see [3–5]).

The description of isomorphisms of classical groups (see the citations in the surveys [11, 17, 31, 33]), Chevalley groups and unitriangular groups over commutative rings (see [1, 6, 15]), as well as the stability of groups over arbitrary rings (see [2]) are very close to the study of homomorphisms of matrix groups over associative rings. Note that the description of stability of linear groups over associative rings (see the citations in the survey [31, 33]) is closely connected with all mentioned above.

The contemporary state of the theory of automorphisms of matrix groups as well as its historical overview is given in the surveys [11, 31, 33] and in the books [12, 17].

2. Preliminaries

In the sequel let R and K be associative rings with 1, let $S \subset K \setminus \{0\}$ be a multiplicatively closed subset such that $1 \in S$ and let K_S be a localization of K by S. Let $\Lambda_S\colon K \to K_S$ be a canonical homomorphism, defined by

$$\Lambda_S(k) = \frac{ks}{s} \qquad (k \in K, s \in S).$$

Let W be a left nonzero K-module and W_S the localization of the module W by S and let $\Lambda\colon GL(n,R) \to GL(W)$ be a group homomorphism. Let $\Lambda_S\colon \mathrm{End}(W) \to \mathrm{End}(W_S)$ be the canonical homomorphism defined by

$$\Lambda_S(\sigma)\left[\frac{w}{s}\right] = \frac{\sigma(w)}{s} \qquad (\sigma \in \mathrm{End}(W), s \in S, w \in W).$$

Clearly Λ_S is a ring homomorphism.

Images of the elements of the rings K and $\mathrm{End}(W)$ as well as the elements of the module W obtained after the localization will be denoted by a bar. Put $\overline{\Lambda} = \Lambda_S\Lambda$. Clearly $\Lambda_S(S) \subseteq K_S^*$, where K_S^* is the group of units of the ring K_S.

Sometime the identity matrix of degree n is denoted by either E_n or 1. Let e_{ij} be a standard matrix unit of the matrix ring $M(n,R)$ over the ring R. The elements

$$t_{ij}(r) = 1 + re_{ij} \quad (i \neq j) \quad \text{and} \quad d_i(r) = 1 + (r-1)e_{ii} \quad (r \in R)$$

are called the elementary transvection and the elementary diagonal matrices of the group $GL(n,R)$, respectively.

The subgroup of $GL(n,R)$ generated by the elementary transvections over a ring R is denoted by $E(n,R)$ and it is called the group of elementary transvections over the ring R.

Definition 1. Let G be a group such that

$$E(n,R) \subseteq G \subseteq GL(n,R) \qquad (n \geq 2)$$

and let $\Lambda\colon G \to GL(W)$ be a homomorphism defined as before.

We say that Λ satisfies *condition* $(*)$, if for any nonzero nilpotent element $m \in \mathrm{End}(W)$ with $m^2 = 0$ (we always assume that such an m exists), the following two items hold:

- there exist $A \in G$ and $s_1 \in \mathbb{N}$ with the property that $s_1 \cdot 1 \in K^*$ such that

$$\Lambda(A) = 1 + s_1 m;$$

- for each $B \in G$ there exists $s_2 \in \mathbb{N}$ with the property that $s_2 \cdot 1 \in K^*$ such that from the equation $\Lambda B \cdot \Lambda A = \Lambda A \cdot \Lambda B$ follows that $BA^{s_2} = A^{s_2}B$.

In particular, condition $(*)$ is satisfied by any isomorphism of the group G to the group $GL(W)$. For this it is enough to assume that $s_1 = s_2 = 1$ and use the fact that $1 + m \in GL(W)$.

Definition 2. Let G be a group such that

$$E(n, R) \subseteq G \subseteq GL(n, R) \qquad (n \geq 2)$$

and let $\Lambda\colon G \to GL(W)$ be a homomorphism defined as before.

We say that Λ satisfies the *extended condition* $(*)$, if for any nonzero nilpotent element $m \in \mathrm{End}(W)$ with $m^2 = 0$ (we always assume that such an m exists), the following two items hold:

- there exist $A \in G$ and $s_1 \in \mathbb{N}$ with the propery that $s_1 \cdot 1 \in K^*$ such that

$$\Lambda(A) = 1 + s_1 m;$$

- for each $B \in G$ there exists $s_2 \in \mathbb{N}$ with the property that $s_2 \cdot 1 \in K^*$ such that from the equation $\Lambda B \cdot \Lambda A = \Lambda A^k \cdot \Lambda B$ follows that $B \cdot A^{s_2} = A^{ks_2} \cdot B$, where k is from a given subset of integers.

In particular, a homomorphism satisfying the extended condition $(*)$ also satisfies condition $(*)$, if the set of integers k in Definition 2 contains 1. For this it is enough to put $k = 1$.

We start with the following.

Lemma 1. *Let R be a ring and let G be a group. If $\Lambda\colon G \to GL(W)$ satisfies the extended condition $(*)$, then the homomorphism $\overline{\Lambda}\colon G \to GL(W_S)$ also satisfies the extended condition $(*)$.*

Proof. Let $\overline{m}$ be a nontrivial nilpotent element of $\mathrm{End}(W_S)$ such that $\overline{m}^2 = \overline{0}$. It follows that $s_0 m \in \mathrm{End}(W)$ for some $s_0 \in S$ and there exists $s_0' \in S$ such that $s_0'(s_0 m)^2 = 0$. Hence $(s_0 s_0' m)^2 = 0$. By the conditions of our definition there exists $A \in G$ such that $\Lambda A = 1 + s_1 m_1$, where $m_1 = s_0 s_0' m$, $s_1 \in S$, $\overline{m}_1 = \overline{m}$ and $\overline{\Lambda} A = \overline{1} + s_1 \overline{m}_1$.

If $g \in G$ and $\overline{\Lambda} g \cdot \overline{\Lambda} A = \overline{\Lambda} A^k \cdot \overline{\Lambda} g$ for some $k \in \mathbb{Z}$, then there exists $s' \in S$, such that $(\Lambda g \cdot \Lambda A - \Lambda A^k \cdot \Lambda g)s' = 0$. Consequently

$$\big(\Lambda g m_1 - k m_1 \Lambda g\big) s_1 s' = 0$$

and, as a consequence $\Lambda g \cdot \Lambda A^{s_1 s'} = \Lambda A^{k s_1 s'} \cdot \Lambda g$. By the extended condition $(*)$ there exists $s_2 \in S$ such that $gA^{s_2} = A^{ks_2} g$.

This proves that $\overline{\Lambda}$ satisfies the extended condition $(*)$. ∎

It is easy to see that a product of a homomorphism which satisfies the extended condition $(*)$ and an isomorphism is also a homomorphism fulfilling the extended condition $(*)$.

We shall use the following result.

Theorem 1. ([24, 25]) *Let R, K be rings with 1 and let G be a group such that*

$$E(n,R) \subseteq G \subseteq GL(n,R) \qquad (n \geq 4).$$

Let W be a left K-module and let $\Lambda\colon G \to GL(W)$ be a homomorphism satisfying the condition $(*)$. Then there exist submodules $L \neq 0$ and P of the module W and an isomorphism $g\colon W \to \underbrace{L \oplus \cdots \oplus L}_{n} \oplus P$ such that

$$\Lambda(x) = g^{-1}\big[\overline{\delta}(x)e + \overline{\nu}(x)^{-1}(1-e) + e_1\big]g.$$

Here $x \in E(n,R)$, $\overline{\delta}\colon M(n,R) \to \mathrm{End}(\underbrace{L \oplus \cdots \oplus L}_{n} \oplus P)$ is a ring homomorphism and $\overline{\nu}\colon M(n,R) \to \mathrm{End}(\underbrace{L \oplus \cdots \oplus L}_{n} \oplus P)$ is a ring antihomomorphism which are induced by the ring homomorphism $\delta\colon R \to \mathrm{End}(L)$ and the ring antihomomorphism $\nu\colon R \to \mathrm{End}(L)$, respectively. The element e is a central idempotent of $\mathrm{End}(L)$, 1 is a unity element of $\mathrm{End}(\underbrace{L \oplus \cdots \oplus L}_{n})$ and e_1 is the unity element of the ring $\mathrm{End}(P)$.

In particular, $\Lambda t_{ij}(1) = g^{-1}[t_{ij}(1)e + t_{ji}(-1)(1-e) + e_1]g$, where $1 \leq i \neq j \leq n$.

Remark 1. If additionally we assume $2 \in R^*$, then the theorem is also true for $n \geq 3$. If $n = 3$ and $2 \notin R^*$ then there exists a nonstandard homomorphism (see [20]).

Let R, K be commutative rings with 1. Let G, G_1 be isomorphic groups with group isomorphism Λ, such that

$$E(n,R) \subseteq G \subseteq GL(n,R), \qquad E(m,K) \subseteq G_1 \subseteq GL(m,K) \quad (n, m \geq 4).$$

Since $\Lambda(G) = G_1 \subseteq GL(m,K) \cong GL(W)$, so Λ satisfies condition $(*)$ by our lemma in Section 2.

Moreover Λ defines the homomorphism

$$\Lambda_I \Lambda\colon E(n,R) \to GL(n,K) \to GL(n,K_I), \tag{1}$$

where I is a maximal ideal of the ring K and K_I is the localization of K by $S = K \backslash I$.

The homomorphism $\Lambda_I\Lambda$ (see (1)) defines the homomorphism

$$\Lambda_0\Lambda\colon E(n,R)\to GL(n,K)\to GL(n,K_0),$$

where $K_0=\prod K_I$ is the direct product of the local rings K_I, where I runs over the maximal ideals of the ring K and also includes the ring K itself. With the help of the description of the homomorphism $\Lambda\colon G\to G_1$ it is easy to describe both $\Lambda_I\Lambda$ and $\Lambda_0\Lambda$. Furthermore, using the property of the homomorphism $\Lambda_0\Lambda$, we can obtain a description of the homomorphism Λ.

Note that the description of the isomorphism $\Lambda\colon G\to G_1$ can be obtained in a different way using ideas of the paper [19]. First of all consider the following commutative diagram:

$$\begin{array}{ccc}
E(n,R) & \xrightarrow{\ \Lambda\ } & GL(m,K) \\
\downarrow & & \downarrow \\
E(n,R_I) & & GL(m,K_J) \\
\downarrow & & \downarrow \\
E(n,R/I) & \xrightarrow{\ \overline{\Lambda}\ } & GL(m,K/J),
\end{array}$$

where $E(n,R)\subseteq G$ and $E(m,K)\subseteq\Lambda(G)$. This is possible due to the fact that the kernel of the homomorphism $E(n,R)\to E(n,R/I)$ for a maximal ideal I of the ring R is normalized by the group $GL(n,R)$ and $\Lambda E(n,R)$ is normalized by the group $\Lambda(G)\supseteq E(m,K)$. Since the structure of the groups which are normalized by $E(m,K)$ over the commutative rings K is known (see [8,12,23,30]), the isomorphism Λ induces the isomorphism of the groups $\overline{\Lambda}\colon E(n,R/I)\to E(m,K/J)$, where J is a maximal ideal of the ring K. Using the fact that the isomorphism between special linear groups over the fields R/I and K/J is also described (see [12]), we have that the fields R/I and K/J are isomorphic, $m=n$ and the homomorphism $\overline{\Lambda}$ is a product of standard homomorphisms. The description of the homomorphisms $\Lambda_I\Lambda$ and $\Lambda_0\Lambda$ follows from which follows the description of the isomorphism Λ.

As a consequence, it is easy to see that in the case for the description of the isomorphism Λ between G and G_1 we need to use a description of the normal structure of matrix groups over the commutative rings and also a description of the isomorphism of matrix groups over the fields, which is not an easy challenge. The proposed approach of this paper does not use these classical results. Rather our results are obtained as a special case of a description of the homomorphisms with condition $(*)$.

3. Main result

The main result of the present paper is the following.

Theorem 2. *Let R, K be commutative rings with 1. Let G, G_1 be isomorphic groups with group isomorphism Λ, such that*

$$E(n,R) \subseteq G \subseteq GL(n,R), \qquad E(m,K) \subseteq G_1 \subseteq GL(m,K) \quad (n,m \geq 4).$$

Then $n = m$ and there exists a ring isomorphism $\delta\colon R \to K$ such that

$$\Lambda t_{ij}(r) = g^{-1}\big[t_{ij}(\delta r)e + t_{ji}(-\delta r)(1-e)\big]g,$$

for some $g \in GL(m, K_0)$. Here $r \in R$, e is an idempotent of the ring R, $1 \leq i \neq j \leq n$ and K_0 is a commutative extension of the ring K.

Proof. Clearly $GL(m,K) \cong GL(W)$ for some left free K-module W of rank m over the ring K. By the conditions of Theorem 1 there exist submodules $L \neq 0$ and P of the module W and there exists an isomorphism of the K-modules

$$g\colon W \to \underbrace{L \oplus \cdots \oplus L}_{n} \oplus P$$

such that

$$\Lambda t_{ij}(r) = g^{-1}\big[t_{ij}(\delta r)e + t_{ji}(-\nu r)(1-e) + e_1\big]g, \tag{2}$$

for all $1 \leq i \neq j \leq n$ and $r \in R$. Here $\delta\colon R \to \mathrm{End}(L)$ is a ring homomorphism, $\nu\colon R \to \mathrm{End}(L)$ is a ring antihomomorphism and e is a central idempotent of the ring $\mathrm{End}(L)$.

Let I be a maximal ideal of the ring K and let $S = K \setminus I$. Let K_I be the localization of the ring K by the multiplicative set S and let W_I be the localization of the module W by S, respectively. From the isomorphism of the left K-modules

$$W \cong \underbrace{L \oplus \cdots \oplus L}_{n} \oplus P$$

follows the isomorphism of the following left K_I-modules

$$W_I \cong \underbrace{L_I \oplus \cdots \oplus L_I}_{n} \oplus P_I.$$

The isomorphism $\Lambda\colon G \to G_1$ induces a homomorphism

$$\Lambda_I \Lambda\colon G \to (G_1)_I \subseteq GL(W_I)$$

which is a homomorphism satisfying condition $(*)$ by the lemma in Section 2 with the condition that $L_I \neq 0$. Since K_I is a commutative local ring and W_I is a free K_I-module of rank m, then the projective submodules L_I and P_I are also free and they have finite rank over the local ring K_I. Using the fact that $L \neq 0$ and $1 \in K$, we obtain that $\operatorname{Ann}_K L \neq K$ and there exists a maximal ideal (which again we call I) of the ring K which contains $\operatorname{Ann}_K L$. Hence

$$\dim(W_I) = m \geq n \dim(L_I) \geq n.$$

Using the same argument for Λ^{-1}, we obtain that $n \geq m$. Consequently $n = m$, $\dim L_I = 1$, $\dim P_I = 0$ so $P_I = 0$. It is easy to see that in the case when $P \neq 0$, there exists a maximal ideal J of the ring K such that $\operatorname{Ann}_K P \subset J$ and $P_J \neq 0$. Consequently $P = 0$ and the homomorphism $\Lambda_I \Lambda$ (see (2)) has the form

$$\Lambda_I \Lambda t_{ij}(r) = \overline{g}^{\,-1} \Big[t_{ij}(\overline{\delta r})\overline{e} + t_{ji}(-\overline{\nu r})(\overline{1} - \overline{e}) \Big] \overline{g},$$

for all $1 \leq i \neq j \leq n$ and $r \in R$. Since K_I is a local ring, it has only trivial idempotents (0 and 1) and $\overline{e} \in \{\overline{0}, \overline{1}\}$ in the ring K_I.

Put $g_I = \overline{g} \in GL(n, K_I)$ and $e_I = \overline{e}$. (Remember that $n = m$.)

The ring K can be written as the cartesian product $K_0 = \prod K_I$ of all local rings K_I, where each K_I is obtained by the localization of K by a different maximal ideal. Clearly the group $GL(n, K)$ can be embedded into the group $GL(n, K_0)$. Put $g_0 = \prod g_I$.

Now put $K_1 = \prod K_I$, $g_1 = \prod g_I$ and $e_1 = \prod e_I$, where in the product we chose those maximal ideals I for which $\overline{e} = \overline{1}$ in K_I. Similarly put $K_2 = \prod K_I$, $g_2 = \prod g_I$ and $e_2 = \prod e_I$ (here we require that $\overline{e} = \overline{0}$).

Clearly $K_0 = K_1 \times K_2$, $GL(n, K_0) = GL(n, K_1) \times GL(n, K_2)$ and $1 = e_1 + e_2$ is the unity element in K and K_0, respectively. This yields that

$$\Lambda t_{ij}(r) = g_0^{\,-1} \big[t_{ij}(\delta r) e_1 + t_{ji}(-\nu r) e_2 \big] g_0 \in G_1 \subseteq GL(n, K), \tag{3}$$

where $\delta \colon R \to K_1$ and $\nu \colon R \to K_2$ are ring homomorphisms such that $\delta(1) = e_1$ and $\nu(1) = e_2$.

For a natural number $k \in \mathbb{N}$ we define $\widehat{k} = \begin{cases} k-1, & \text{if } k \text{ is even;} \\ k+1, & \text{if } k \text{ is odd.} \end{cases}$

Put

$$C = \Big[e_1 E_n + e_2 \prod t_{k\widehat{k}}(1) t_{\widehat{k}k}(-1) t_{k\widehat{k}}(1) \Big] g_0,$$

where k is even and $1 \le k, \widehat{k} \le n$. Now it is easy to check (see (3)) that

$$\Lambda t_{k\widehat{k}}(1) = C^{-1} t_{k\widehat{k}}(1) C \in M(n, K);$$
$$\Lambda t_{ij}(1) = C^{-1}\big[t_{ij}(1)e_1 + t_{\widehat{j}\widehat{i}}(-1)^{i+j} e_2\big] C \in M(n, K);$$
$$\Lambda t_{ij}(1) - \Lambda t_{\widehat{j}\widehat{i}}(-1)^{i+j} = C^{-1}\big[e_{ij} + (-1)^{i+j+1} e_{\widehat{j}\widehat{i}}\big] C \in M(n, K)$$

for all $1 \le i \ne \widehat{j} \le n$.

If n is odd, then we need to consider additionally the elements $\Lambda t_{in}(1)$, $\Lambda t_{ni}(1)$ and the following difference:

$$\Lambda t_{in}(1) - \Lambda t_{n\widehat{i}}(-1)^{\widehat{i}} \in M(n, K).$$

Put $C^{-1} = (c_{ij}) \in M(n, K_0)$ and $C = (C_{ij}) \in M(n, K_0)$. It is easy to check that the elements of the rows of the matrix C and the elements of the columns of C^{-1} can be expressed as a linear combination over K in terms of each other, respectively. Hence $(c_{1i} \cdots c_{ni})^{\mathrm{T}}(C_{j1} \cdots C_{jn})$ is a matrix over the ring K (here T is the classical transpose). This proves that $c_{ki}C_{jl} \in K$ for all $1 \le k, i, j, l \le n$. Since K_0 is a commutative ring, $C_{ki}c_{jl} \in K$ for all $1 \le k, i, j, l \le n$. Hence

$$t_{ij}(1)e_1 + t_{\widehat{j}\widehat{i}}(-1)^{i+j} e_2 = C \Lambda t_{ij}(1) C^{-1} \in GL(n, K).$$

Moreover, e_1, e_2 are idempotents of the ring K and $C^{-1}GL(n, K)C = GL(n, K)$. Put $\delta_0 = \delta + \nu$. Then $e_1 \nu r = e_2 \delta r = 0$ and

$$\Lambda t_{ij}(r) = g_0{}^{-1}\big[t_{ij}(\delta_0 r)e_1 + t_{ji}(-\delta_0 r)e_2\big] g_0,$$

for all $1 \le i \ne j \le n$ and $\delta_0 \colon R \to K$ is the isomorphism. ∎

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V. M. Petechuk, Transcarpathian Regional Institute of Postgraduate Pedagogical Education, Uzhgorod, A. Voloshyna Str. 35, Uzhgorod, Ukraine; *e-mail*: Vasyl.M.Petechuk@gmail.com

J. V. Petechuk, Uzhgorod National University, Uzhgorod, Ukraine; *e-mail*: Julia.V.Petechuk@gmail.com